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Note

Characterisation of self-complementary chordal graphs

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Abstract

In this note we prove that a self-complementary graph with p vertices is chordal if and only if its clique number is integral part of $(p+1)/2$. © 1998 Elsevier Science B.V. All rights reserved

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1. Preliminaries

We refer to [4] for the basic definitions not mentioned here. The integral part of a real number r is denoted by $\lfloor r \rfloor$. For a graph G let p, V, $[v,w]$, E and $\bar{G}$ denote the number of vertices, the vertex set, the edge incident with the vertices v and w, the edge set and the complement of G respectively. Let v be a vertex of G and B be a subset of V not containing the vertex v. We define $[v,B]$ to be the set of all edges incident with v and a vertex in B. The clique number $\omega(G)$ is the maximum number of vertices of G that can induce a complete subgraph of G and a complete subgraph of G with $\omega(G)$ vertices is called a maximum clique of G. A stable set S of G is a subset of V such that no two of the vertices in S are adjacent in G. The degree sequence of G denoted by $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ is the non-increasing sequence of the degrees of the vertices of G. A chordal graph is a graph which has no induced cycle with n vertices for all $n \geqslant 4$. A split graph is a graph whose vertex set can be partitioned into V' and S such that the induced subgraph $\langle V' \rangle$ is a complete subgraph and S is a stable set of the graph.

A graph is self-complementary (s.c.) if it is isomorphic to its complement. For a survey on s.c. graphs we refer to [1]. Every s.c. graph has $4n$ or $4n+1$ vertices. A

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complementing permutation (c.p.) σ of a s.c. graph G is a vertex isomorphism of G onto $\bar{G}$. Let G be a s.c. graph with a c.p. $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ where σ_i's are the disjoint cycles such that $\sigma_i = (v_{i1}v_{i2}\cdots v_{ip_i})$. If a σ of G has a single cycle $(v_{11}v_{12}\cdots v_{1p})$ we identify the vertices v_{1j} as j where $1 \leqslant j \leqslant p$. The length of a cycle σ_i is the number of vertices present in σ_i. A cycle σ_i is a k-cycle if its length is k. A vertex v is present in σ_i is denoted by $v \in \sigma_i$. A vertex of σ_i is even or odd labelled if the subscript j of v_{ij} is even or odd respectively. The set consisting of all even labelled vertices and the set consisting of all odd labelled vertices of σ_i are denoted by $\mathrm{Even}(\sigma_i)$ and $\mathrm{Odd}(\sigma_i)$ respectively. We define $\mathrm{Even}(\sigma) = \bigcup_{i=1}^{s} \mathrm{Even}(\sigma_i)$ and $\mathrm{Odd}(\sigma) = \bigcup_{i=1}^{s} \mathrm{Odd}(\sigma_i)$, respectively. Let $B \subseteq V$. The set of all vertices $\sigma(v)$ such that $v \in B$ is denoted by $\sigma(B)$. By σ^k we mean σ multiplied k times.

2. Basic results

Theorem 1 (Foldes and Hammer [2]). *Let G be a graph. Then G is a split graph if and only if G and $\bar{G}$ are chordal graphs.*

We note the following as a consequence of the above theorem.

Theorem 2. *Let G be a s.c. graph. Then G is a split graph if and only if it is a chordal graph.*

Theorem 3 (Hammer and Simone [3]). *Let G be a graph with degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$. Let $m = \max\{i : d_i \geqslant i-1,\ 1 \leqslant i \leqslant p\}$. Then G is a split graph implies $\omega(G)=m$.*

Let G be a s.c. graph with a c.p. σ. Ringel [5] and Sachs [6] proved that the length of every cycle of σ is divisible by 4 with a possible exception of a single cycle of length 1 in case G has $4n+1$ vertices. Let u and v be two vertices of G. Then $[u,v] \in E$ if and only if $[\sigma^{2i}(u), \sigma^{2i}(v)] \in E$ and $[\sigma^{2i+1}(u), \sigma^{2i+1}(v)] \notin E$ for every integer i.

Definition. Let G be a s.c. graph with $p = 4n$. A c.p. $\sigma^* = \sigma_1^*\sigma_2^*\cdots\sigma_s^*$ of G where $\sigma_i^* = (v_{i1}v_{i2}\cdots v_{ip_i})$ such that $[v_{i2}, v_{i4}] \in E$ is called a star c.p. of G.

3. Main results

Lemma 1. *Let G be a s.c. graph with degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$. Let $m = \max\{i : d_i \geqslant i-1,\ 1 \leqslant i \leqslant p\}$. Then $m = \lfloor (p+1)/2 \rfloor$.*

Proof. The proof follows from the fact that $d_i + d_{p-i+1} = p-1$ for all $1 \leqslant i \leqslant p$. $\square$

Theorem 4. *Let G be a s.c. graph. Then $\omega(G) = \lfloor (p+1)/2 \rfloor$ if G is a chordal graph.*

Proof. Follows from Lemma 1, and Theorems 2 and 3. □

Lemma 2. *Let G be a s.c. graph. Then $\omega(G) \leqslant \lfloor (p+1)/2 \rfloor$.*

Proof. Let $\omega(G) \geqslant \lfloor (p+1)/2 \rfloor + 1$. Then there exists a complete subgraph K of G with $\lfloor (p+1)/2 \rfloor + 1$ vertices. Let V_K be the vertex set of K. Also there exists a subgraph S of G isomorphic to $\bar{K}$ since G is a s.c. graph. We note that S is a stable set of G. So K and S can have atmost one common vertex. This implies $V_K \cup S$ has atleast $2\lfloor (p+1)/2 \rfloor + 1$ elements, a contradiction. Therefore $\omega(G) \leqslant \lfloor (p+1)/2 \rfloor$. □

Lemma 3. *Let G be a s.c. graph with $p = 4n$ and a c.p. σ. Let V' be a subset of V such that $\langle V' \rangle$ is a complete subgraph of G. If $v \in V'$ and $\sigma(v) \in V'$ for some vertex v of G then either $w \notin V'$ or $\sigma(w) \notin V'$ for every vertex w of G distinct from v.*

Proof. The proof is straightforward. □

Lemma 4. *Let G be a s.c. graph with $p = 4n(\geqslant 8)$, $\omega(G) = 2n$, a maximum clique C and a c.p. $\sigma = (1\ 2 \cdots 4n)$. Let V' be the vertex set of C. There exists no vertex i such that both $i \in V'$ and $\sigma(i) \in V'$ where $1 \leqslant i \leqslant 4n$.*

Proof. Let $k \in V'$ and $\sigma(k) \in V'$ for some vertex $1 \leqslant k \leqslant 4n$. We note that $\sigma^2(k) \notin V'$ and $\sigma^{4n-1}(k) \notin V'$ since $[\sigma(k), \sigma^2(k)] \notin E$ and $[\sigma^{4n-1}(k), \sigma^{4n}(k)] = [\sigma^{4n-1}(k), k] \notin E$. Also either $\sigma^3(k) \notin V'$ or $\sigma^{4n-2}(k) \notin V'$ since $[\sigma(k), \sigma^3(k)] \notin E$ or $[\sigma^{4n-2}(k), \sigma^{4n}(k)] = [\sigma^{4n-2}(k), k] \notin E$. By Lemma 3 at most one vertex of each set $\{\sigma^4(k), \sigma^5(k)\}, \{\sigma^6(k), \sigma^7(k)\}, \ldots, \{\sigma^{4n-4}(k), \sigma^{4n-3}(k)\}$ belongs to V'. Hence, either $\sigma^3(k)$ or $\sigma^{4n-2}(k)$ but not both and exactly one vertex of each set $\{\sigma^4(k), \sigma^5(k)\}, \{\sigma^6(k), \sigma^7(k)\}, \ldots, \{\sigma^{4n-4}(k), \sigma^{4n-3}(k)\}$ belongs to V'. Let $\sigma^3(k) \in V'$. Then $\sigma^5(k), \sigma^7(k), \ldots, \sigma^{4n-3}(k)$ belong to V' since $[\sigma^3(k), \sigma^4(k)] \notin E, [\sigma^5(k), \sigma^6(k)] \notin E, \ldots, [\sigma^{4n-5}(k), \sigma^{4n-4}(k)] \notin E$. But either $\sigma^3(k) \notin V'$ or $\sigma^{4n-3}(k) \notin V'$ since $[k, \sigma^3(k)] \notin E$ or $[\sigma^{4n-3}(k), \sigma^{4n}(k)] = [\sigma^{4n-3}(k), k] \notin E$, a contradiction. So $\sigma^3(k) \notin V'$. This implies $\sigma^{4n-2}(k) \in V'$. Proceeding similarly as in the previous case leads to a contradiction. Therefore the lemma. □

Lemma 5. *Let G be a s.c. graph with $p = 4n(\geqslant 8)$, $\omega(G) = 2n$ and a star c.p. $\sigma^* = (1\ 2 \cdots 4n)$. Then $\langle \mathrm{Even}(\sigma^*) \rangle$ is the unique maximum clique of G.*

Proof. Follows from Lemma 4. □

Lemma 6. *Let G be a s.c. graph with $p = 4n$, $\omega(G) = 2n$ and a star c.p. σ^*. Then $\langle \mathrm{Even}(\sigma^*) \rangle$ is a maximum clique of G.*

Proof. If $p = 4$ the result is trivial. Let $p \geqslant 8$. Let $\sigma^* = \sigma_1^* \sigma_2^* \cdots \sigma_s^*$. Let the length of the cycle σ_i^* be p_i. Let $V_i = \{v \in \sigma_i^*\}$ where $1 \leqslant i \leqslant s$. Let C be a maximum clique of G. We note that C has $p_i/2$ vertices from every V_i. Otherwise it would lead to a contradiction to Lemma 2. By Lemma 5 C has all even labelled vertices of σ_i^*'s whose length is atleast 8 since $\langle V_i \rangle$ is a s.c. graph with a c.p. σ_i^*. Also C has both an odd labelled and an even labelled vertex from atmost one 4-cycle of σ^*. Otherwise we get a contradiction to Lemma 3 since for every 4-cycle σ_i^* either $\sigma_i^*(v) = w$ or $\sigma_i^*(w) = v$ where v and w are odd labelled and even labelled vertices of σ_i^* respectively. Clearly, C cannot have the two odd labelled vertices of any 4-cycle of σ^*. Hence, C has only even labelled vertices of σ^* with a possible exception of a single odd labelled vertex of a 4-cycle σ_m^*. If C has only even labelled vertices of σ^* we are done. Otherwise an odd labelled vertex v and an even labelled vertex w of a 4-cycle σ_m^* of σ^* belong to C. Let $B = \{v \in \mathrm{Even}(\sigma_i^*) : 1 \leqslant i \leqslant s,\ i \neq m\}$. We note that $[w, B] \subseteq E$. This implies $[\sigma^{*^2}(w), B] \subseteq E$ since $\sigma^{*^2}(B) = B$. Also $[w, \sigma^{*^2}(w)] \in E$ since $\sigma^{*^2}(w)$ is an even labelled vertex of σ_m^* distinct from w. Hence $\langle \mathrm{Even}(\sigma^*) \rangle$ is a maximum clique of G since $\mathrm{Even}(\sigma^*) = B \cup \{w, \sigma^{*^2}(w)\}$. □

Theorem 5. *Let G be a s.c. graph with $p=4n$ and $\omega(G) = 2n$. Then G is a split graph.*

Proof. Let σ^* be a star c.p. of G. By Lemma 6 $\langle \mathrm{Even}(\sigma^*) \rangle$ is a complete subgraph of G. We note that $\mathrm{Odd}(\sigma^*)$ is a stable set since $\mathrm{Odd}(\sigma^*) = \sigma^*(\mathrm{Even}(\sigma^*))$. The sets $\mathrm{Even}(\sigma^*)$ and $\mathrm{Odd}(\sigma^*)$ partition V. Hence, G is a split graph. □

Theorem 6. *Let G be a s.c. graph with $p = 4n + 1$ and $\omega(G) = 2n + 1$. Then G is a split graph.*

Proof. Let σ be a c.p. of G. Let σ_m be the 1-cycle of σ. Let $v_0 \in \sigma_m$. Let G' be the graph obtained from G by deleting v_0. Clearly G' is a s.c. graph. Let C be a maximum clique of G and let V_C be the vertex set of C. The vertex v_0 belongs to V_C. Otherwise $\omega(G') \geqslant 2n+1$, a contradiction to Lemma 2. Let $S = \sigma(V_C)$. We note that S is a stable set of G. Also $v_0 \in S$, since $\sigma(v_0) = v_0$. The sets S and V_C have atmost one common element since S is a stable set of G and $\langle V_C \rangle$ is a complete subgraph of G. Hence V_C and $S - \{v_0\}$ partition V since S and V_C has $2n + 1$ elements. Therefore G is a split graph. □

Theorem 7. *Let G be a s.c. graph. Then G is a chordal graph if and only if $\omega(G) = \lfloor (p+1)/2 \rfloor$.*

Proof. Follows from Theorems 2 and 4–6. □

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